

Corréction L2 SPS
Examen 05/01/2016

(1)

$$\begin{aligned} \underline{\text{Exercice 1}} \quad \forall x \neq 0 \quad \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x} &= \frac{1+x^2 - (1-x^2)}{x \times (\sqrt{1+x^2} + \sqrt{1-x^2})} = \frac{2x}{x(\sqrt{1+x^2} + \sqrt{1-x^2})} \\ &= \frac{2}{\sqrt{1+x^2} + \sqrt{1-x^2}} \quad \xrightarrow{x \rightarrow 0} \frac{2}{2} = 1 \end{aligned}$$

$$\frac{\exp(x+x^2)-1}{x} = \frac{\exp(x+x^2)-1}{x+x^2} \times \frac{x+x^2}{x} = \frac{\exp(x+x^2)-1}{x+x^2} \times (1+x)$$

Sachant que $\lim_{u \rightarrow 0} \frac{\exp(u)-1}{u} = 1$, $x+x^2 \rightarrow 0$ quand $x \rightarrow 0$

on obtient $\lim_{x \rightarrow 0} \frac{\exp(x+x^2)-1}{x} = 1$

Par les croissances comparées, il vient $\lim_{x \rightarrow +\infty} \frac{(\ln x)^{2016}}{\sqrt{x}} = 0$

$$\begin{aligned} \underline{\text{Exercice 2}} \quad \int_2^4 \underbrace{x^2 \ln(2x)}_{u'} dx &= \left[\frac{x^3}{3} \ln(2x) \right]_2^4 - \int_2^4 \frac{x^3}{3} \times \frac{1}{x} dx \\ &= \frac{64}{3} \ln 8 - \frac{8}{3} \ln 4 - \int_2^4 \frac{x^2}{3} dx \\ &= \frac{3 \times 64 - 2 \times 8}{3} \ln 2 - \left[\frac{x^3}{9} \right]_2^4 = \frac{176}{3} \ln 2 - \left[\frac{64}{9} - \frac{8}{9} \right] = \frac{176}{3} \ln 2 - \frac{56}{9} \end{aligned}$$

$$\begin{aligned} \int_0^2 \underbrace{(x+1)}_{u'} \underbrace{\exp(3x)}_{v'} dx &= \left[(x+1) \frac{\exp(3x)}{3} \right]_0^2 - \int_0^2 1 \times \frac{\exp(3x)}{3} dx \\ &= \frac{2}{3} \exp(6) - \frac{1}{3} - \left[\frac{\exp(3x)}{9} \right]_0^2 = \exp(6) - \frac{1}{3} - \frac{\exp 6}{9} + \frac{1}{9} \\ &= \frac{8}{9} \exp(6) - \frac{2}{9} \end{aligned}$$

$$\begin{aligned} \underline{\text{Exercice 3.}} \quad f_1(x) &= \cos(\exp(x^2)) \quad f'_1(x) = -\sin(\exp(x^2)) \times \exp(x^2) \times 2x \\ f_2(x) &= \frac{\ln x}{x^4+1} \quad \Rightarrow \quad f'_2(x) = \frac{\frac{x^4+1}{x} - 4x^3 \ln x}{(x^4+1)^2} = \frac{x^4+1 - 4x^4 \ln x}{x (x^4+1)^2} \end{aligned}$$

$$\begin{aligned} \underline{\text{Exercice 4.}} \quad f(x) &= \frac{1}{2} (\exp(x) - \exp(-x)). \quad \left\{ \begin{array}{l} \exp(-x) > 0 \\ \exp(x) > 0 \quad \forall x \in \mathbb{R} \end{array} \right. \text{ on obtient} \\ i) \quad f'(x) &= \frac{1}{2} (\exp(x) + \exp(-x)). \quad \text{Comme } \exp(x) > 0 \quad \forall x \in \mathbb{R} \end{aligned}$$

Exercice 4 (suite)

ii) $\lim_{x \rightarrow +\infty} \exp(x) = +\infty$, $\lim_{x \rightarrow +\infty} \exp(-x) = \lim_{x \rightarrow +\infty} \frac{1}{\exp(x)} = \frac{1}{+\infty} = 0$.

donc $\lim_{x \rightarrow +\infty} f(x) = +\infty$. De même on trouve $\lim_{x \rightarrow -\infty} f(x) = -\infty$.

iii) f est une fonction dérivable sur \mathbb{R} , strictement croissante sur \mathbb{R} . Elle définit donc une bijection de \mathbb{R} dans $\left] \lim_{x \rightarrow -\infty} f(x), \lim_{x \rightarrow +\infty} f(x) \right[$, soit une bijection de \mathbb{R} dans \mathbb{R} .

iv) $f(x) = y$ si $\frac{1}{2}(e^x - e^{-x}) = y$. On peut déjà vérifier que $x = \ln(y + \sqrt{y^2 + 1})$ est une solution.

$$\begin{aligned} & \frac{1}{2}(\exp(\ln(y + \sqrt{y^2 + 1})) - \exp(-\ln(y + \sqrt{y^2 + 1}))) \quad \text{par définition} \\ &= \frac{1}{2} \left(\cancel{y + \sqrt{y^2 + 1}} - \cancel{-y - \sqrt{y^2 + 1}} \right) \\ &= \frac{1}{2} \left(\exp(\ln(y + \sqrt{y^2 + 1})) - \exp(\ln(\frac{1}{y + \sqrt{y^2 + 1}})) \right) \\ &= \frac{1}{2} \left(y + \sqrt{y^2 + 1} - \frac{1}{y + \sqrt{y^2 + 1}} \right) = \frac{1}{2} \left(\frac{y^2 + 2y\sqrt{y^2 + 1} + y^2 + 1 - 1}{y + \sqrt{y^2 + 1}} \right) \\ &= \frac{1}{2} \left(\frac{2y(y + \sqrt{y^2 + 1})}{y + \sqrt{y^2 + 1}} \right) = y \quad \blacksquare \end{aligned}$$

D'après la question iii) f étant une bijection la solution x de $f(x) = y$ est unique. Donc $f(x) = y$ si $x = \ln(y + \sqrt{y^2 + 1})$

v) $g(x) = \ln(x + \sqrt{x^2 + 1})$ $g'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \times \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right)$

$$g'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \times \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \right) = \frac{1}{\sqrt{x^2 + 1}}$$

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Examen 05/01/2016

(3)

Exercice 5.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} = A^2$$

$$\begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & 0 & 0 \end{pmatrix} = A^3$$

Pour calculer A^2 on pose le produit

$$A - 3A^2 + A^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} - 3 \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 8 & -1 \\ 8 & 13 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6+5 & 1-9+8 & 1-1 \\ 1-9+8 & 2-15+13 & -1+5-2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercice 6.

$$\left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ 3x + 2y + 3z - t = -2 \\ -2x + y + 3z + t = -3 \\ 3x - y - z - 2t = 0 \end{array} \right.$$

$$\begin{array}{l} l_2 \leftarrow l_2 + \frac{3}{2}l_1 \\ l_3 \leftarrow l_3 - l_1 \\ l_4 \leftarrow l_4 + \frac{3}{2}l_1 \end{array}$$

$$\left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ 0 + \frac{13}{2}y + \frac{15}{2}z + \frac{7}{2}t = -\frac{1}{2} \\ 0 - 2y + 0z - 2t = -4 \\ 0 + \frac{7}{2}y + \frac{7}{2}z + \frac{5}{2}t = \frac{3}{2} \end{array} \right.$$

$$\left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ 13y + 15z + 7t = -1 \\ -y - t = -2 \\ 7y + 7z + 5t = 3 \end{array} \right.$$

on choisit l_3 pour éliminer y en l_2 et l_4

$$\left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ 15z - 6t = -27 \\ -y - t = -2 \\ 7z - 2t = -11 \end{array} \right.$$

$$\begin{array}{l} l_2 \leftarrow l_2 + 13l_3 \\ l_4 \leftarrow l_4 + 7l_3 \end{array}$$

$$\left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ -y - t = -2 \\ 15z - 6t = -27 \quad l_3 \leftarrow \frac{l_3}{3} \\ 7z - 2t = 11 \end{array} \right. \quad \left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ -y - t = -2 \\ 5z - 2t = -9 \\ 7z - 2t = 11 \end{array} \right.$$

$$l_4 \leftarrow l_4 - l_3 \quad \left\{ \begin{array}{l} -2x + 3y + 3z + 3t = 1 \\ -y - t = -2 \\ 5z - 2t = -9 \\ 2z = -2 \end{array} \right. \quad \begin{array}{l} x = 1 \\ y = 0 \\ z = 2 \\ t = -1 \end{array}$$

Exercise 7

$$\left(\begin{array}{ccc} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 2 & 1 & -5/2 \end{array} \right) \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

$$l_2 \leftarrow l_2 - 2l_1$$

$$l_3 \leftarrow l_3 - 2l_1$$

$$\left(\begin{array}{ccc} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & -1 & -1/2 \end{array} \right) \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{array} \right)$$

$$l_1 \leftarrow l_1 + \frac{l_2}{2} \quad \left(\begin{array}{ccc} 1 & 0 & 1/2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{array} \right) \quad \left(\begin{array}{ccc} 0 & 1/2 & 0 \\ -2 & 1 & 0 \\ -1 & -1/2 & 1 \end{array} \right)$$

$$l_1 \leftarrow l_1 + \frac{l_3}{4}$$

$$l_2 \leftarrow l_2 + \frac{3}{2}l_3 \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{array} \right) \quad \left(\begin{array}{ccc} -1/4 & 3/8 & 1/4 \\ -7/2 & 3/4 & 3/2 \\ -1 & -1/2 & 1 \end{array} \right)$$

$$l_2 \leftarrow \frac{l_2}{-2}$$

$$l_3 \leftarrow \frac{l_3}{-2}$$

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(5)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 3/8 & 1/4 \\ 7/4 & -1/8 & -3/4 \\ 1/2 & +1/4 & -1/2 \end{pmatrix} = A^{-1}$$